On localization of pseudo-relativistic energy

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We present a Kato-type inequality for bounded domains $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$.

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1. Introduction

Hardy's inequality is an important tool in the study of the spectral properties of partial differential equations. This inequality states that for a function $f \in C_0^\infty(\mathbb{R}^n)$, $n \geqslant 3$

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2}\, dx \leqslant \text{const.} \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx.$$

The corresponding "first order" analogue of the Hardy inequality was established by Kato and plays an important role in the study of relativistic quantum mechanical systems. Specifically, Kato inequality states that for $f \in C_0^\infty(\mathbb{R}^n)$, $n \geqslant 2$

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|}\, dx \leqslant \text{const.} \int_{\mathbb{R}^n} (\sqrt{-\Delta} f(x), f(x))\, dx, \qquad (1.1)$$

where $\Delta = \sum_{k=1}^{n} \partial_k^2$.

The analogue of Hardy's inequality for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ with a Lipshitz boundary is

$$\int_{\Omega} \frac{|u(x)|^2}{(\rho_\Omega(x))^2}\, dx \leqslant \text{const.} \int_{\Omega} |\nabla u(x)|^2\, dx,$$

where $\rho_\Omega(x) = \min_{x_0 \in \partial\Omega} |x - x_0|$ (Edmunds&Evans (2004) p. 212; see also Davies (1984, 1999) and Lewis (1988) for references and details).

The purpose of this article is to establish the Kato-type inequality for a bounded domain $\Omega \subset \mathbb{R}^n$. Since $\sqrt{-\Delta}$ is a non-local operator, there are three possibilities to define the r.h.s. of (1.1) in the case of $\Omega \subset \mathbb{R}^n$. One possibility is to use the r.h.s. of (1.1) but restrict ourselves only to functions with compact support inside Ω. Another possibility is based on the fact that (see Lieb&Loss (1997))

$$\int_{\mathbb{R}^n} (\sqrt{-\Delta} f(x), f(x))\, dx = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}}\, dx dy.$$

So we can define the analogue of the r.h.s. of (1.1) for Ω as

$$\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+1}}\, dx dy\,. \qquad (1.2)$$

The third possibility is to consider square root of the internal Dirichlet Laplacian operator in the domain Ω.

Inthis article we consider the first two definitions, since they are more interesting for relativistic quantum mechanics (localization of kinetic energy). The case of a Kato-type inequality for the square root of the internal Dirichlet-Lapalcian in fact follows for nice domains from Hardy's inequality since

$$A^2 \geqslant B^2 \Longrightarrow A \geqslant B$$

for operators $A, B > 0$ (see Birman&Solomjak (1987), Theorem 2, p. 232).

Let us briefly describe the content of the paper. In section 2 for functions f such that $supp\, f \subset \Omega_1$ for some $\overline{\Omega}_1 \subset \Omega$ we show that

$$\int\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+1}}\, dx dy \geqslant \mathrm{const.} \int\limits_{\Omega_1} \frac{|f(x)|^2}{\rho_{\Omega_1}(x)}\, dx\,, \qquad (1.3)$$

where $\rho_{\Omega_1}(x)$ is the distance from x to $\partial\Omega_1$, i.e. $\rho_{\Omega_1}(x) = \min\limits_{z\in\partial\Omega_1} |z-x|$. Later we obtain the inequality

$$\int\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+1}}\, dx dy \geqslant \mathrm{const.} \int\limits_{\Omega} \frac{|f(x)|^2}{\rho_\Omega(x)(1+|\ln\rho_\Omega(x)|^3)}\, dx\,. \qquad (1.4)$$

Initially we prove (1.4) for radial functions (Proposition 1) and then for all $f \in L^2(\Omega, \mathbb{C})$ (Theorem 2). Though we give (1.4) for some restricted class of bounded domains Ω we expect that Theorem 2 is true for more general domains. But we will not discuss this in the current article.

2. Kato-type inequality for functions with compact support.

Theorem 1. *Let Ω_1 be a convex bounded domain such that $\overline{\Omega}_1 \subset \Omega$ for some domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. We suppose that $f \in L^2(\Omega, \mathbb{C}^1)$ and $supp\, f \subset \Omega_1$. Then for some constant $c_1 = c_1(\Omega, \Omega_1) > 0$ the inequality (1.3) holds.*

In view of the inequality $|f(x)-f(y)| \geqslant ||f(x)|-|f(y)||$, without loss of generality we may assume that $f(x)$ is a real-valued function. Next we apply the Lieb-Yau trick (see Lieb&Yau (1988)) to get inequality which is a basic tool in the proofs of Theorems 1 and 2.

Lemma 1. *Let $K : B \times B \to \mathbb{R}$, $h : B \to \mathbb{R}$, where $B \subset \mathbb{R}^m$, $m \in \mathbb{N}$. We assume that $K \in L^\infty(B \times B)$,*

$$K(x,y) = K(y,x)\,, \qquad K(x,y) \geqslant 0 \qquad (2.1)$$

and

$$0 < M < h(z) < M^{-1} \qquad (2.2)$$

for any $x, y, z \in B$ and some constant $M > 0$. Then

$$\int\limits_B \int\limits_B (f(x) - f(y))^2 \, K(x,y) \, dxdy \geqslant 2 \int\limits_B f^2(x) \left[\int\limits_B K(x,y) \left(1 - \frac{h(x)}{h(y)} \right) dy \right] dx \quad (2.3)$$

for any $f \in L^2(B)$ with bounded support.

Proof. On expanding brackets in the l.h.s. of (2.3) we get

$$\int\limits_B \int\limits_B (f(x) - f(y))^2 \, K(x,y) \, dxdy \qquad (2.4)$$

$$\geqslant \quad 2 \int\limits_B f^2(x) \left[\int\limits_B K(x,y) \, dy \right] dx - 2 \int\limits_B \int\limits_B f(x)f(y) \, K(x,y) \, dxdy \, .$$

Applying the Cauchy-Schwartz inequality and using (2.1), (2.2) gives

$$\int\limits_B \int\limits_B f(x)f(y) \, K(x,y) \, dxdy$$

$$= \quad \int\limits_B \int\limits_B \left(f(x) \sqrt{\frac{K(x,y)h(x)}{h(y)}} \right) \left(f(y) \sqrt{\frac{K(y,x)h(y)}{h(x)}} \right) dxdy$$

$$\leqslant \quad \int\limits_B \int\limits_B f^2(x) \, K(x,y) \frac{h(x)}{h(y)} \, dxdy \, . \qquad (2.5)$$

The inequality (2.3) follows from (2.4) and (2.5). □

Corollary 1. *Let us suppose that $K(x,y)$ and $f(x)$ satisfy to conditions of Lemma 1 and $\operatorname{supp} f \subset \Omega_1$ for some $\Omega_1 \subset B$. Then*

$$\int\limits_B \int\limits_B (f(x) - f(y))^2 \, K(x,y) \, dxdy \geqslant 2 \int\limits_{\Omega_1} f^2(x) \left[\int\limits_{B\setminus\Omega_1} K(x,y) \, dy \right] dx \qquad (2.6)$$

Proof. An application of Lemma 1 with

$$h_\varepsilon(z) = \begin{cases} 1 & \text{if} \quad z \in \Omega_1 \\ 1/\varepsilon & \text{otherwise} \end{cases}$$

gives

$$\int\limits_B \int\limits_B (f(x) - f(y))^2 \, K(x,y) \, dxdy \geqslant 2 \int\limits_{\Omega_1} f^2(x) L_\varepsilon(x) \, dx \, ,$$

where

$$L_\varepsilon(x) = \int\limits_B K(x,y) \left(1 - \frac{h_\varepsilon(x)}{h_\varepsilon(y)} \right) dy = \int\limits_{B\setminus\Omega} K(x,y)(1-\varepsilon) \, dy \, .$$

Passing to the limit $\varepsilon \to 0$ completes the proof. □

Proof of Theorem 1. In view of Corollary 1 it suffices to prove that

$$\int\limits_{\Omega\backslash\Omega_1} \frac{dy}{|x-y|^{n+1}} \geqslant \frac{c_2}{\rho(x)} \tag{2.7}$$

for any $x \in \Omega_1$ and some $c_2 = c_2(\Omega, \Omega_1) > 0$. The convexity of Ω_1 implies that for any $z \in \partial\Omega_1$ there exists an $(n-1)$-dimensional plane π_z in $\mathbb{R}^n$ such that $z \in \pi_z$ and $\pi_z \cap \Omega_1 = \emptyset$. For any $x \in \Omega_1$ we take $x_0 = x_0(x)$ such that $\rho_{\Omega_1}(x) = |x - x_0|$. Let D_{x_0} be the half of $\mathbb{R}^n$ with boundary π_{x_0} which does not contain Ω_1. Clearly

$$\int\limits_{\Omega\backslash\Omega_1} \frac{dy}{|x-y|^{n+1}} \geqslant \int\limits_{\Omega\cap D_{x_0}} \frac{dy}{|x-y|^{n+1}}.$$

For any $z \in \partial\Omega_1$ we put

$$\kappa_1(z) := \sup\{s > 0 : B_s(z) \subset \Omega\}, \qquad \kappa := \inf_{z\in\partial\Omega_1} k_1(z),$$

where $B_s(z)$ is a ball with center at z and radius s. From $\overline{\Omega}_1 \subset \Omega$ we conclude that $\kappa = \kappa(\Omega, \Omega_1) > 0$. Consequently we have

$$\int\limits_{\Omega\cap D_{x_0}} \frac{dy}{|x-y|^{n+1}} \geqslant \int\limits_{B_\kappa(x_0)\cap D_{x_0}} \frac{dy}{|x-y|^{n+1}}. \tag{2.8}$$

Let us choose Cartesian coordinates $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$ with center at x_0 and axes such that $D_{x_0} = \{y : y_1 \geqslant 0\}$. Then $x = (-\rho_{\Omega_1}(x), 0)$ and $B_\kappa(x_0) = \{y : y_1^2 + \ldots + y_n^2 \leqslant \kappa^2\}$. Making the change of variables $y_1 = \rho_{\Omega_1}(x)z_1, \ldots, y_n = \rho_{\Omega_1}(x)z_n$ in (2.8) gives

$$\int\limits_{B_\kappa(x_0)\cap D_{x_0}} \frac{dy}{|x-y|^{n+1}} = \frac{1}{\rho_{\Omega_1}(x)} \int\limits_{S_1} \frac{dz_1 \ldots dz_n}{((z_1+1)^2 + z_2^2 + \ldots + z_n)^{\frac{n+1}{2}}},$$

where

$$S_1 = \left\{z = (z_1, \ldots, z_n) : z_1 > 0 \quad \text{and} \quad z_1^2 + \ldots + z_n^2 \leqslant \kappa^2(\rho_{\Omega_1}(x))^{-2}\right\}.$$

Since Ω_1 is bounded, it follows that for some constant $c_3 = c_3(\Omega_1) > 0$

$$\rho_{\Omega_1}(x) \leqslant c_3$$

for all $x \in \Omega_1$. Therefore $\rho_{\Omega_1}^{-2}(x) \geqslant c_3^{-2}$ and so

$$S_2 := \left\{z = (z_1, \ldots, z_n) : z_1 > 0 \quad \text{and} \quad z_1^2 + \ldots + z_n^2 \leqslant \kappa^2 c_3^{-2}\right\} \subset S_1.$$

Combining the above estimates we obtain (2.7) with

$$c_2 = \int\limits_{S_2} \frac{dz_1 \ldots dz_n}{((z_1+1)^2 + z_2 + \ldots + z_n^2)^{\frac{n+1}{2}}}.$$

$\square$

3. Lower estimate for the integral representation (1.2). Case of radial functions.

Proposition 1. *We suppose that $f : \mathbb{R}^1 \to \mathbb{R}^1$ and $\operatorname{supp} f \subset [0,1)$. Then for some absolute constant $c_4 > 0$ we get*

$$\int\limits_{B_1(0) \times B_1(0)} \frac{(f(|x|) - f(|y|))^2}{|x - y|^{n+1}} \, dx \, dy \geqslant c_4 \int\limits_{B_1(0)} \frac{(f(|x|))^2}{(1 - |x|)(1 - (\ln(1 - |x|))^3)} \, dx \,,$$

(3.1)

where $B_1(0) \subset \mathbb{R}^n$, $n \geqslant 2$, is a ball with center at the origin and radius $R = 1$.

Let us briefly outline the content of this section. The proof of Proposition 1 is preceded by proofs of some auxiliary results. In Lemma 2 we show that integral on the l.h.s. of (3.1) is equivalent (up to multiplication by a constant) to one-dimensional integral (3.3). In order to estimate (3.3) from below we apply the Lieb-Yau trick (Lemma 1) with test function

$$h(r) = 100 - (1 - r)^\omega$$

for $\omega \in (0, 1/4)$ and then integrate in ω both sides of the obtained inequality. Lemmas 3, 4 are needed to get lower estimate for the term $\int_B K(x,y)(1 - h(x)/h(y)) \, dy$ on the r.h.s. of (2.3). At the end of this section we piece together all the lemmas to establish Proposition 1.

Lemma 2. *Under the conditions of Proposition 1, for some constant $c_5 = c_5(n) > 0$ we have*

$$c_5 \, \mathbf{I} \leqslant \int\limits_{B_1(0) \times B_1(0)} \frac{(f(|x|) - f(|y|))^2}{|x - y|^{n+1}} \, dx \, dy \leqslant 2^{3-2n} \pi^{2n-3} c_5 \, \mathbf{I} \,,$$

(3.2)

where

$$\mathbf{I} = \int\limits_0^1 \int\limits_0^1 \frac{(f(r) - f(s))^2}{(r - s)^2} \left(\frac{rs}{r + s} \right)^{n-1} dr \, ds \,.$$

(3.3)

Proof. Let us change the coordinates x, y in the integral in (3.2) to spherical coordinates $x = (r, \theta_1, \ldots, \theta_{n-1})$, $y = (s, \phi_1, \ldots, \phi_{n-1})$, where

$$r, s \in [0, 1], \qquad \theta_1, \ldots, \theta_{n-2}, \phi_1, \ldots, \phi_{n-2} \in [0, \pi], \qquad \theta_{n-1}, \phi_{n-1} \in [0, 2\pi) \,.$$

We choose the direction of the axes in y-space such that the direction of axis $\phi_1 = \pi/2$ coincides with the vector x, i.e the angle between x and y is equal to ϕ_1 and so

$$|x - y|^2 = |x|^2 + |y|^2 - 2|x||y| \cos \phi_1 \,.$$

Recall that the absolute value of the Jacobian of this change of variables is equal to

$$\left(r^{n-1} |\sin \theta_1|^{n-2} \ldots |\sin \theta_{n-2}|^1 \right) \times \left(s^{n-1} |\sin \phi_1|^{n-2} \ldots |\sin \phi_{n-2}|^1 \right) \,.$$

It follows that

$$\int\limits_{B_1(0)\times B_1(0)} \frac{(f(|x|)-f(|y|))^2}{|x-y|^{n+1}}\, dxdy$$

$$=c_6 \int\limits_0^1 \int\limits_0^1 \int\limits_0^\pi \frac{(f(r)-f(s))^2}{(r^2+s^2-2rs\cos\phi_1)^{\frac{n+1}{2}}}\, (rs)^{n-1}|\sin\phi_1|^{n-2}\, drdsd\phi_1\,,$$

where

$$c_6 = \int\limits_0^\pi |\sin\theta_1|^{n-2}\, d\theta_1 \left(\int\limits_0^\pi |\sin\theta_2|^{n-3}\, d\theta_2\right)^2 \times \ldots \times \left(\int\limits_0^\pi |\sin\theta_{n-2}|\, d\theta_{n-2}\right)^2.$$

Denote by $J(k)$ the Euler-type integral

$$J(k) := \int\limits_0^\pi \frac{|\sin\phi|^{n-2}d\phi}{(k^2+\sin^2(\phi/2))^{\frac{n+1}{2}}} = 2\int\limits_0^{\frac{\pi}{2}} \frac{|\sin(2\phi)|^{n-2}d\phi}{(k^2+\sin^2\phi)^{\frac{n+1}{2}}}\,. \qquad (3.4)$$

Then using

$$r^2+s^2-2rs\cos\phi_1 = (r-s)^2 + 4rs\sin^2(\phi_1/2)$$

we obtain

$$\int\limits_{B_1(0)\times B_1(0)} \frac{(f(|x|)-f(|y|))^2}{|x-y|^{n+1}}\, dxdy$$

$$= c_6 \int\limits_0^1 \int\limits_0^1 \frac{(f(r)-f(s))^2(rs)^{n-1}}{(4rs)^{\frac{n+1}{2}}}\, J\left(\frac{|r-s|}{2\sqrt{rs}}\right)\, drds\,. \qquad (3.5)$$

From (3.4) and the elementary inequality $2|z|/\pi \leqslant |\sin z| \leqslant |z|$ for $z \in [-\pi/2, \pi/2]$ we find that

$$2^{n-1}\left(\frac{2}{\pi}\right)^{n-2}\int\limits_0^{\frac{\pi}{2}} \frac{\phi^{n-2}d\phi}{(k^2+\phi^2)^{\frac{n+1}{2}}} \leqslant J(k) \leqslant 2^{n-1}\int\limits_0^{\frac{\pi}{2}} \frac{\phi^{n-2}d\phi}{(k^2+(2/\pi)^2\phi^2)^{\frac{n+1}{2}}}\,.$$

Since

$$\int \frac{\phi^{n-2}}{(1+\phi^2)^{\frac{n+1}{2}}}\, d\phi = \frac{\phi^{n-1}}{(n-1)(1+\phi^2)^{\frac{n-1}{2}}} + \text{const.}\,,$$

it follows that

$$J(k) \geqslant \frac{1}{k^2}\, 2^{n-1}\left(\frac{2}{\pi}\right)^{n-2} \frac{(\pi/2k)^{n-1}}{(n-1)(1+(\pi/2k)^2)^{\frac{n-1}{2}}}\,,$$

$$J(k) \leqslant \frac{1}{k^2}\, 2^{n-1}\left(\frac{\pi}{2}\right)^{n-1} \frac{(1/k)^{n-1}}{(n-1)(1+(1/k)^2)^{\frac{n-1}{2}}}$$

or

$$\frac{1}{k^2} \frac{2^{n-2}\pi}{(n-1)(k^2 + (\pi/2)^2)^{\frac{n-1}{2}}} \leqslant J(k) \leqslant \frac{1}{k^2} \frac{\pi^{n-1}}{(n-1)(k^2+1)^{\frac{n-1}{2}}}.$$

An application of the elementary inequality

$$k^2 + \left(\frac{\pi}{2}\right)^2 \leqslant \left(\frac{\pi}{2}\right)^2 (k^2 + 1)$$

implies that

$$\frac{1}{k^2} \frac{2^{2n-3}\pi^{2-\pi}}{(n-1)(k^2+1)^{\frac{n-1}{2}}} \leqslant J(k) \leqslant \frac{1}{k^2} \frac{\pi^{n-1}}{(n-1)(k^2+1)^{\frac{n-1}{2}}}.$$

Hence

$$J\left(\frac{|r-s|}{2\sqrt{rs}}\right) \leqslant \frac{4rs}{(r-s)^2} \frac{\pi^{n-1}(2\sqrt{rs})^{n-1}}{(n-1)(r+s)^{n-1}} = \frac{2^{n+1}\pi^{n-1}(rs)^{\frac{n+1}{2}}}{(n-1)(r+s)^{n-1}(r-s)^2}$$

and

$$J\left(\frac{|r-s|}{2\sqrt{rs}}\right) \geqslant \frac{4rs}{(r-s)^2} \frac{2^{2n-3}\pi^{2-n}(2\sqrt{rs})^{n-1}}{(n-1)(r+s)^{\frac{n-1}{2}}} = \frac{2^{3n-2}\pi^{2-n}(rs)^{\frac{n+1}{2}}}{(n-1)(r+s)^{n-1}(r-s)^2}.$$

Substituting these estimates into (3.5) we obtain

$$\frac{2^{2n-3}\pi^{2-n}}{(n-1)} c_6 \, \mathbf{I} \leqslant \int_{B_1(0) \times B_1(0)} \frac{(f(|x|) - f(|y|))^2}{|x-y|^{n+1}} \, dx dy \leqslant \frac{\pi^{n-1}}{(n-1)} c_6 \, \mathbf{I}.$$

Taking $c_5 = 2^{2n-3}\pi^{2-n}c_6/(n-1)$ we arrive at (3.2). □

Lemma 3. *Let $\phi(\cdot)$ be a positive increasing function and*

$$h(r) = \phi((1-r)^{-1}). \qquad (3.6)$$

Then for any $r \in (0,1)$

$$r^{-(n-1)} \lim_{\varepsilon \to 0} \int_0^1 \frac{(\min\{r,s\})^{n-1}}{\varepsilon^2 + (r-s)^2} \left(1 - \frac{h(r)}{h(s)}\right) ds$$

$$\geqslant \mu \lim_{\varepsilon \to 0} \int_{\mu^{-1}}^{+\infty} \frac{1}{\varepsilon^2 u^2 + (u-1)^2} \left(\frac{\phi(\mu u) - \phi(\mu)}{\phi(\mu u)}\right) du, \qquad (3.7)$$

where $\mu = (1-r)^{-1}$, $n \geqslant 2$.

Proof. Step 1. We have

$$\int_0^1 (\min\{r,s\})^{n-1} I \, ds = \int_0^r s^{n-1} I \, ds + r^{n-1} \int_r^1 I \, ds$$

$$= r^{n-1} \int_0^1 I \, ds + \int_0^r (s^{n-1} - r^{n-1}) I \, ds,$$

where

$$I = \frac{1}{\varepsilon^2 + (r-s)^2}\left(1 - \frac{h(r)}{h(s)}\right).$$

Since $h(s)$ is increasing, then $I < 0$ for $s < r$, and so

$$\int_0^r (s^{n-1} - r^{n-1})I\,ds \geqslant 0.$$

Thus

$$\text{l.h.s. of } (3.7) = \lim_{\varepsilon \to 0} r^{-(n-1)} \int_0^1 (\min\{r,s\})^{n-1} I\,ds$$

$$\geqslant \lim_{\varepsilon \to 0} \int_0^1 \frac{1}{\varepsilon^2 + (r-s)^2}\left(1 - \frac{h(r)}{h(s)}\right) ds. \tag{3.8}$$

Step 2. Let us make the change of the variables

$$u = \frac{1-r}{1-s} \tag{3.9}$$

in the integral on the r.h.s of (3.8). Elementary calculations give

$$s = r + (1-r)\left(1 - \frac{1}{u}\right),$$

$$\frac{1}{1-s} = \frac{u}{1-r}, \tag{3.10}$$

$$\frac{1}{\varepsilon^2 + (r-s)^2} = \frac{u^2}{\varepsilon^2 u^2 + (1-r)^2(u-1)^2}, \tag{3.11}$$

$$ds = \frac{1-r}{u^2}\,du \tag{3.12}$$

and

$$0 \leqslant s < 1 \qquad \Leftrightarrow \qquad 1-r \leqslant u < +\infty. \tag{3.13}$$

Consequently, using (3.6) and (3.9)-(3.13) we get

$$\text{r.h.s. of } (3.8) = \lim_{\varepsilon \to 0} \int_{1-r}^\infty \frac{1-r}{\varepsilon^2 u^2 + (1-r)^2(u-1)^2}\left(1 - \frac{\phi\left(\frac{1}{1-r}\right)}{\phi\left(\frac{u}{1-r}\right)}\right) du$$

$$= \frac{1}{1-r}\lim_{\varepsilon \to 0} \int_{1-r}^\infty \frac{1}{(1-r)^{-2}\varepsilon^2 u^2 + (u-1)^2}\left(1 - \frac{\phi\left(\frac{1}{1-r}\right)}{\phi\left(\frac{u}{1-r}\right)}\right) du. \tag{3.14}$$

Substituting $\mu = (1-r)^{-1}$ into (3.14) and making the change $\varepsilon := \mu^2 \varepsilon$ we arrive at

$$\text{r.h.s. of } (3.8) \geqslant \mu \lim_{\varepsilon \to 0} \int_{\mu^{-1}}^{+\infty} \frac{1}{\varepsilon^2 u^2 + (u-1)^2}\left(\frac{\phi(\mu u) - \phi(\mu)}{\phi(\mu u)}\right) du. \tag{3.15}$$

Combining (3.8) and (3.15) completes the proof. □

Lemma 4. *There exist absolute constants $c_7 > 0$ and $\kappa > 0$ such that for any $0 < \omega < 1/4$ and $\mu > 1$*

$$\lim_{\varepsilon \to 0} \int_{\mu^{-1}}^{+\infty} \frac{1}{\varepsilon^2 u^2 + (u-1)^2} \left(\frac{\phi(\mu u) - \phi(\mu)}{\phi(\mu u)} \right) du \geqslant \frac{c_7 \, \omega^2}{\mu^\omega}, \tag{3.16}$$

where

$$\phi(z) = \kappa - z^{-\omega}. \tag{3.17}$$

Proof. Substituting (3.17) into the l.h.s of (3.16) and using

$$\frac{1}{(\kappa - \mu^{-\omega} u^{-\omega})} = \frac{1}{(\kappa - \mu^{-\omega})} + \frac{\mu^{-\omega} u^{-\omega} - \mu^{-\omega}}{(\kappa - \mu^{-\omega})(\kappa - \mu^{-\omega} u^{-\omega})}$$

we get

$$\text{l.h.s. of (3.16)} = \lim_{\varepsilon \to 0} \int_{\mu^{-1}}^{\infty} \frac{\mu^{-\omega} - \mu^{-\omega} u^{-\omega}}{(\varepsilon^2 u^2 + (u-1)^2)(\kappa - \mu^{-\omega} u^{-\omega})} du$$

$$= \frac{\mu^{-\omega}}{\kappa - \mu^{-\omega}} A - \frac{\mu^{-2\omega}}{\kappa - \mu^{-\omega}} B \tag{3.18}$$

with

$$A(\mu) = \lim_{\varepsilon \to 0} \int_{\mu^{-1}}^{+\infty} \frac{1 - u^{-\omega}}{\varepsilon^2 u^2 + (u-1)^2} du$$

and

$$B(\mu) = \lim_{\varepsilon \to 0} \int_{\mu^{-1}}^{+\infty} \frac{(u^{-\omega} - 1)^2}{(\varepsilon^2 u^2 + (u-1)^2)(\kappa - \mu^{-\omega} u^{-\omega})} du$$

$$= \int_{\mu^{-1}}^{+\infty} \frac{(u^{-\omega} - 1)^2}{(u-1)^2 (\kappa - \mu^{-\omega} u^{-\omega})} du. \tag{3.19}$$

Let us estimate $A(\mu)$ and $B(\mu)$. Since $\mu^{-1} \leqslant 1$ and $1 - u^{-\omega} < 0$ for $u < 1$, it follows that

$$A(\mu) \geqslant \lim_{\varepsilon \to 0} \int_{0}^{+\infty} \frac{1 - u^{-\omega}}{\varepsilon^2 u^2 + (u-1)^2} du. \tag{3.20}$$

For any $R \in (1, +\infty)$ we put

$$\gamma_R = \{ u \in \mathbb{C} : |u| = R, \, \text{Im} \, u > 0 \}, \qquad \gamma_R^1 = [0, R], \qquad \gamma_R^2 = [-R, 0].$$

Let γ_R be oriented anticlockwise and segments γ_R^1, γ_R^2 oriented from left to right. Due to the fact that for any $\varepsilon < 1$

$$\int_{\gamma_R} \frac{1 - u^{-\omega}}{\varepsilon^2 u^2 + (u-1)^2} du \to 0 \quad \text{as} \quad R \to +\infty$$

an application of Cauchy's theorem gives

$$\int\limits_0^{+\infty} \frac{1 - u^{-\omega}}{\varepsilon^2 u^2 + (u-1)^2}\, du = \lim_{R \to +\infty} \int\limits_{\gamma_R^2} \frac{1 - u^{-\omega}}{\varepsilon^2 u^2 + (u-1)^2}\, du = \int\limits_0^{+\infty} \frac{1 - t^{-\omega} e^{-\mathrm{i}\pi\omega}}{\varepsilon^2 t^2 + (t+1)^2}\, dt\,.$$

$$(3.21)$$

Combining (3.20) and (3.21) we have

$$A(\mu) \geqslant \int\limits_0^{+\infty} \frac{1 - \cos(\pi\omega) t^{-\omega}}{(t+1)^2}\, dt = (1 - \cos(\pi\omega)) \int\limits_0^{+\infty} \frac{dt}{(t+1)^2} - \cos(\pi\omega)\psi(\omega)\,,$$

where

$$\psi(\omega) = \int\limits_0^{+\infty} \frac{t^{-\omega} - 1}{(t+1)^2}\, dt\,.$$

Using the elementary inequality

$$1 - \cos\alpha \geqslant \frac{\alpha^2}{4} \qquad \text{for} \qquad \alpha \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$$

we get

$$A(\mu) \geqslant \frac{(\pi\omega)^2}{4} - \psi(\omega)\,.$$

$$(3.22)$$

Making the change of the variables $t = z^{-1}$ we get

$$\int\limits_0^{+\infty} \frac{(\ln t)^{2m-1}}{(t+1)^2}\, dt = -\int\limits_0^{+\infty} \frac{(\ln z)^{2m-1}}{(z+1)^2}\, dz$$

and so

$$\psi^{(2m-1)}(0) = \int\limits_0^{+\infty} \frac{(\ln t)^{2m-1}}{(t+1)^2}\, dt = 0 \qquad \text{for all} \quad m \in \mathbb{N}\,.$$

$$(3.23)$$

Moreover,

$$\psi^{(2m)}(0) = \int\limits_0^{+\infty} \frac{(\ln t)^{2m}}{(t+1)^2}\, dt > 0 \qquad \text{for all} \quad m \in \mathbb{N}\,.$$

$$(3.24)$$

Applying (3.23), (3.24) to Taylor's expansion of $\psi(\omega)$

$$\frac{\psi(\omega)}{\omega^2} = \sum_{m=1}^\infty \frac{\psi^{(2m)}(\omega)}{(2m)!}\, \omega^{2(m-1)}$$

we see that $\psi(\omega)\omega^{-2}$ increases for $\omega > 0$ and so

$$\frac{\psi(\omega)}{\omega^2} \leqslant 16\,\psi\left(\frac{1}{4}\right)$$

for $0 < \omega < 1/4$. Therefore, by (3.22) we have

$$A(\mu) \geqslant \left[\frac{\pi^2}{4} - 16\,\psi\left(\frac{1}{4}\right)\right]\omega^2 = c_8\omega^2\,, \tag{3.25}$$

where

$$c_8 = \frac{\pi^2}{4} - 16 \int\limits_0^{+\infty} \frac{t^{-\frac{1}{4}} - 1}{(t+1)^2}\,dt \approx 0.695869349\,. \tag{3.26}$$

We proceed with $B(\mu)$. According to (3.19)

$$B(\mu) = \int\limits_{\mu^{-1}}^{+\infty} \frac{(u^{-\omega} - 1)^2}{(u-1)^2(\kappa - \mu^{-\omega}u^{-\omega})}\,du \leqslant \frac{1}{\kappa - 1} \int\limits_0^{+\infty} \frac{(u^{-\omega} - 1)^2}{(u-1)^2}\,du\,.$$

Since

$$(u^{-\omega} - 1)^2 \leqslant \omega^2(\ln u)^2(u^{2\omega} + u^{-2\omega}) \leqslant \omega^2(\ln u)^2(u^{\frac{1}{2}} + u^{-\frac{1}{2}})$$

for all $0 < \omega < 1/4$ and all $u > 0$, it follows that

$$B(\mu) \leqslant \frac{c_9}{\kappa - 1}\,\omega^2\,, \tag{3.27}$$

where

$$c_9 = \int\limits_0^{+\infty} \frac{(\ln u)^2(u^{\frac{1}{2}} + u^{-\frac{1}{2}})}{(u-1)^2}\,du \approx 39.47841761\,. \tag{3.28}$$

Combining (3.18) with (3.25), (3.27) we have

$$\text{l.h.s. of (3.16)} = \frac{\mu^{-\omega}}{\kappa - \mu^{-\omega}}\left(A - \mu^{-\omega}B\right) \geqslant \frac{\mu^{-\omega}\omega^2}{\kappa - \mu^{-\omega}}\left(c_8 - \mu^{-\omega}\frac{c_9}{\kappa - 1}\right)$$

$$\geqslant \frac{\mu^{-\omega}\omega^2}{\kappa + 1}\left(c_8 - \frac{c_9}{\kappa - 1}\right)$$

for all $\mu > 1$ and $0 < \omega < 1/4$. Taking $\kappa = 100$ and using (3.26), (3.28) we obtain (3.16) with

$$c_7 = \frac{1}{\kappa + 1}\left(c_8 - \frac{c_9}{\kappa - 1}\right) \approx 0.002941558950\,.$$

$\square$

Lemma 5. *One has*

$$\int\limits_0^{\frac{1}{4}} \frac{\omega^2}{\mu^\omega}\,d\omega \geqslant \frac{c_{10}}{1 + (\ln\mu)^3}$$

for all $\mu \geqslant 1$ and some absolute constant $c_{10} > 0$.

Proof. After elementary calculations we get

$$\psi(\mu) := \int_0^{\frac{1}{4}} \frac{\omega^2}{\mu^\omega}\, d\omega = -\frac{2 + 2\omega \ln\mu + \omega^2 (\ln\mu)^2}{(\ln\mu)^3 \mu^\omega}\bigg|_0^{\frac{1}{4}} = \frac{2 - \delta(\mu)}{(\ln\mu)^3}, \qquad (3.29)$$

where

$$\delta(\mu) := \frac{2 + 2^{-1}\ln(\mu) + 4^{-2}(\ln\mu)^2}{\mu^{\frac{1}{4}}}.$$

Since

$$\delta'(\mu) = -\frac{(\ln\mu)^2}{64\mu^{\frac{5}{4}}} < 0,$$

and so $2 - \delta(\mu) \geqslant 2 - \delta(e)$ for $\mu \geqslant e$, it follows that

$$\psi(\mu) \geqslant \frac{2 - \delta(e)}{(\ln\mu)^3} \geqslant \frac{2 - \delta(e)}{1 + (\ln\mu)^3}$$

for $\mu \geqslant e$. On the other hand, since $\psi'(\mu) = -\psi(\mu)(\ln\mu) < 0$ we get

$$\psi(\mu) \geqslant \psi(e) \geqslant \frac{\psi(e)}{1 + (\ln\mu)^3}$$

for $1 \leqslant \mu \leqslant e$. Note that, by (3.29), $\psi(e) = 2 - \delta(e)$. Taking

$$c_{10} = 2 - \delta(e) = 2 - \frac{41}{16\, e^{\frac{1}{4}}} \approx 0.004322994$$

we complete the proof.   □

Proof of Proposition 1. Using the left inequality in (3.2) and the fact that

$$\frac{rs}{r+s} \geqslant \frac{1}{2}\, \min\{r, s\}$$

we find

$$\int_{B_1(0)\times B_1(0)} \frac{(f(|x|) - f(|y|))^2}{|x - y|^{n+1}}\, dxdy$$

$$\geqslant \quad \frac{c_5}{2^{n-1}} \int_0^1 \int_0^1 \frac{(f(r) - f(s))^2}{(r - s)^2}\, (\min\{r, s\})^{n-1}\, drds$$

$$= \quad \frac{c_5}{2^{n-1}} \lim_{\varepsilon\to 0} \int_0^1 \int_0^1 \frac{(f(r) - f(s))^2}{\varepsilon^2 + (r - s)^2}\, (\min\{r, s\})^{n-1}\, drds. \quad (3.30)$$

An application of Lemma 1 with $m = 1$, $B = [0, 1]$,

$$K(r, s) = \frac{(\min\{r, s\})^{n-1}}{\varepsilon^2 + (r - s)^2}$$

for any positive function $h(\cdot)$ gives

$$\int\limits_0^1 \int\limits_0^1 \frac{(f(r) - f(s))^2}{\varepsilon^2 + (r - s)^2} \left(\min\{r, s\}\right)^{n-1} dr ds \geqslant \int\limits_0^1 (f(r))^2 L_\varepsilon(r)\, dr\,, \qquad (3.31)$$

where

$$L_\varepsilon(r) = 2 \int\limits_0^1 \frac{(\min\{r, s\})^{n-1}}{(\varepsilon^2 + (r - s)^2)} \left(1 - \frac{h(r)}{h(s)}\right) ds\,.$$

Let $h(\cdot)$ and $\phi(\cdot)$ be defined by (3.6) and (3.17) respectively. An application of Lemmas 3 and 4 yields

$$\lim_{\varepsilon \to 0} L_\varepsilon(r) \geqslant 2r^{n-1}\mu \lim_{\varepsilon \to 0} \int\limits_{\mu^{-1}}^{+\infty} \frac{1}{\varepsilon^2 u^2 + (u - 1)^2} \left(\frac{\phi(\mu u) - \phi(\mu)}{\phi(\mu u)}\right) du$$

$$\geqslant \frac{2c_7 r^{n-1}\mu\,\omega^2}{\mu^\omega}\,, \qquad (3.32)$$

where as before $\mu = (1 - r)^{-1}$. Combining (3.30)-(3.32) we obtain

$$\int\limits_{B_1(0) \times B_1(0)} \frac{(f(|x|) - f(|y|))^2}{|x - y|^{n+1}}\, dx dy \geqslant \frac{c_5 c_7}{2^{n-2}} \int\limits_0^1 \frac{r^{n-1}\mu\,\omega^2}{\mu^\omega}\, dr. \qquad (3.33)$$

Integrating both sides of (3.33) in ω and using Lemma 5 we have

$$\int\limits_0^{\frac{1}{4}} 1\, d\omega \int\limits_{B_1(0) \times B_1(0)} \frac{(f(|x|) - f(|y|))^2}{|x - y|^{n+1}}\, dx dy$$

$$\geqslant \frac{c_5 c_7}{2^{n-2}} \int\limits_0^1 \frac{r^{n-1}(f(r))^2}{1 - r} \left(\int\limits_0^{\frac{1}{4}} \frac{\omega^2}{\mu^\omega}\, d\omega\right) dr$$

$$\geqslant \frac{c_5 c_7 c_{10}}{2^{n-2}} \int\limits_0^1 \frac{r^{n-1}(f(r))^2}{(1 - r)(1 - (\ln(1 - r))^3)}\, dr.$$

We put

$$c_4 = \frac{4 c_5 c_7 c_{10}}{2^{n-2} c_{11}}\,,$$

where

$$c_{11} = \int\limits_0^\pi |\sin\theta_1|^{n-2}\, d\theta_1 \int\limits_0^\pi |\sin\theta_2|^{n-3}\, d\theta_2 \times \ldots \times \int\limits_0^\pi |\sin\theta_{n-2}|\, d\theta_{n-2}\,. \qquad (3.34)$$

Thus

$$\int\limits_{B_1(0)\times B_1(0)} \frac{(f(|x|)-f(|y|))^2}{|x-y|^{n+1}}\, dxdy \geqslant c_4 c_{11} \int\limits_0^1 \frac{r^{n-1}(f(r))^2}{(1-r)(1-(\ln(1-r))^3)}\, dr\,. \quad (3.35)$$

Let us substitute (3.34) into (3.35) and change the variables $(r,\theta_1,\ldots,\theta_{n-1})$ on the r.h.s. of (3.35) to Cartesian coordinates. Then (3.1) follows. $\qquad\square$

4. Lower estimate for the integral representation (1.2). General case.

Here we generalize inequality (3.1) to the case of non-radial functions. Furthermore we obtain the analogue of (3.1) for certain class of domains Ω.

Lemma 6. *Let* $f \in L^2\left(\mathbb{R}^n,\mathbb{C}^1\right)$, $n \geqslant 2$ *such that* $supp\, f \subset B_1(0)$. *Then*

$$\int\limits_{B_1(0)\times B_1(0)} \frac{|f(x)-f(y)|^2}{|x-y|^{n+1}}\, dxdy \geqslant c_4 \int\limits_{B_1(0)} \frac{|f(x)|^2}{(1-|x|)(1-(\ln(1-|x|))^3)}\, dx\,, \quad (4.1)$$

where $c_4 > 0$ *is the absolute constant from Proposition 1.*

Proof. In view of the inequality $|f(x)-f(y)| \geqslant ||f(x)|-|f(y)||$, without loss of generality we may assume that $f(x)$ is real-valued.

For any $e \in S^n$ (S^n is the unit sphere in $\mathbb{R}^n$) we put

$$T^e : \mathbb{R}^n \to \mathbb{R}^n\,, \qquad T^e z = 2(z,e)e - z\,,$$

i.e. T^e is rotation in $\mathbb{R}^n$ around e through angle π. Obviously

$$|T^e x - T^e y| = |x-y| \qquad (4.2)$$

for all $x, y \in \mathbb{R}^n$.

Making the change of the variables $x := T^e x$, $y := T^e y$ and using (4.2) and $|\det T^e| = 1$ we have

$$\int\limits_{B_1(0)\times B_1(0)} \frac{|f(x)-f(y)|^2}{|x-y|^{n+1}}\, dxdy = \int\limits_{B_1(0)\times B_1(0)} \frac{|f(T^e x)-f(T^e y)|^2}{|x-y|^{n+1}}\, dxdy\,. \quad (4.3)$$

According to the Cauchy-Schwartz inequality

$$\int\limits_{S^n} f(T^e x) f(T^e y)\, de \leqslant \left(\int\limits_{S^n} |f(T^e x)|^2\, de\right)^{\frac{1}{2}} \left(\int\limits_{S^n} |f(T^e y)|^2\, de\right)^{\frac{1}{2}}$$

and so

$$\int\limits_{S^n} |f(T^e x)-f(T^e y)|^2\, de$$

$$= \int\limits_{S^n} |f(T^e x)|^2\, de + \int\limits_{S^n} |f(T^e y)|^2\, de - 2\int\limits_{S^n} f(T^e x) f(T^e y)\, de$$

$$\geqslant (\psi(x)-\psi(y))^2\,,$$

where

$$\psi(x) := \left(\int\limits_{S^n} |f(T^e x)|^2 \, de \right)^{\frac{1}{2}} .$$

Using this and integrating (4.3) over all $e \in S^n$ gives

$$|S^n| \int\limits_{B_1(0) \times B_1(0)} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} \, dxdy \geqslant \int\limits_{B_1(0) \times B_1(0)} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{n+1}} \, dxdy . \quad (4.4)$$

Note that $\psi(x)$ depends only on $|x|$. Hence we can apply Proposition 1. It follows that

$$\int\limits_{B_1(0) \times B_1(0)} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{n+1}} \, dxdy \geqslant c_4 \int\limits_{B_1(0)} \frac{|\psi(x)|^2}{(1 - |x|)(1 - (\ln(1 - |x|))^3)} \, dx$$

$$= c_4 \int\limits_{B_1(0)} \frac{\int\limits_{S^n} |f(T^e x)|^2 \, de}{(1 - |x|)(1 - (\ln(1 - |x|))^3)} \, dx .$$

Since

$$\int\limits_{B_1(0)} \frac{|f(T^e x)|^2}{(1 - |x|)(1 - (\ln(1 - |x|))^3)} \, dx = \int\limits_{B_1(0)} \frac{|f(x)|^2}{(1 - |x|)(1 - (\ln(1 - |x|))^3)} \, dx$$

for any $e \in S^n$, it follows that

$$\int\limits_{B_1(0) \times B_1(0)} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{n+1}} \, dxdy \geqslant c_4 |S^n| \int\limits_{B_1(0)} \frac{|f(x)|^2}{(1 - |x|)(1 - (\ln(1 - |x|))^3)} \, dx .$$

$$(4.5)$$

Combining (4.4), (4.5) we complete the proof. $\square$

Theorem 2. *Let Ω be a domain in $\mathbb{R}^n$, $n \geqslant 2$. We assume that there exist diffeo-morphism*

$$\phi : B_1(0) \to \Omega$$

and some constant $c_{12} = c_{12}(\Omega) > 1$ such that for all $u \in B_1(0)$ $\nabla \phi(u) > 0$ and

$$c_{12}^{-1} \leqslant \lambda_i(u) \leqslant c_{12} \qquad i = 1, \ldots, n , \quad (4.6)$$

where $\lambda_i(u)$ are eigenvalues of the matrix $\nabla \phi(u)$. Then for some constant $c_{14} = c_{14}(\Omega) > 0$ and any $f \in L^2(\Omega, \mathbb{C}^1)$ we have

$$\int\limits_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} \, dxdy \geqslant c_{14} \int\limits_{\Omega} \frac{|f(x)|^2}{\rho_\Omega(x)(1 + |\ln \rho_\Omega(x)|^3)} \, dx ,$$

where $\rho_\Omega(y) := \min\limits_{y_0 \in \partial \Omega} |y - y_0|$.

Proof. Step 1. (4.6) and the fact that $\det \nabla \phi(u) = \lambda_1 \ldots \lambda_n$ imply that

$$c_{12}^{-n} \leqslant |\det \nabla \phi(u)| \leqslant c_{12}^n, \qquad c_{12}^{-n} \leqslant \left|\det \nabla \phi^{-1}(x)\right| \leqslant c_{12}^n, \qquad (4.7)$$

for all $u \in B_1(0)$ and $x \in \Omega$. Moreover, for all $u, v \in B_1(0)$ an application the mean value theorem to $\psi_1(\tau) = \phi(\tau u + (1 - \tau)v)$, $\tau \in [0, 1]$ gives

$$|\phi(u) - \phi(v)| = |\psi_1(1) - \psi_1(0)| \leqslant \max_{\tau \in [0,1]} |\psi_1'(\tau)| \leqslant c_{12}|u - v|, \qquad (4.8)$$

where we have used, by (4.6)

$$|\psi_1'(\tau)| \leqslant \max_{w \in B_1(0)} \|\nabla \phi(w)\| \, |u - v| \leqslant \max_{w \in B_1(0)} \max_{i=1..n} \lambda_i(w) \, |u - v| \leqslant c_{12}|u - v|.$$

Similarly, for all $u, v \in B_1(0)$ we obtain

$$|u - v| = |\psi_2(1) - \psi_2(0)| \leqslant \max_{\tau \in [0,1]} |\psi_2'(\tau)| \leqslant c_{12}|\phi(u) - \phi(v)|, \qquad (4.9)$$

where $\psi_2(\tau) = \phi^{-1}(\tau \phi(u) + (1 - \tau)\phi(v))$.

As before we put $\rho_D(x) = \min_{z \in \partial D} |x - z|$ for any domain D. Given any $x \in \Omega$ we put $u = \phi^{-1}(x)$ and take u_0, x_1 are such that

$$\rho_{B_1(0)}(u) = |u_0 - u|, \qquad \rho_\Omega(x) = |x_1 - x|,$$

i.e. u_0, x_1 deliver minima to corresponding functionals. Applying (4.8) we get

$$\rho_{B_1(0)}(u) = |u_0 - u| \geqslant c_{12}^{-1}|\phi(u_0) - \phi(u)| = c_{12}^{-1}|x_0 - x|,$$

where $x_0 = \phi(u_0)$. By choice of x_1 we have $|x_0 - x| \geqslant |x_1 - x|$ and so

$$\rho_{B_1(0)}(u) \geqslant c_{12}^{-1}|x_1 - x| = c_{12}^{-1}\rho_\Omega(x). \qquad (4.10)$$

Along these lines using (4.9) we get

$$\rho_\Omega(x) \geqslant c_{12}^{-1}\rho_{B_1(0)}(u). \qquad (4.11)$$

Step 2. Making the change of variables $x = \phi(u)$, $y = \phi(v)$, using (4.7)-(4.8) and letting $g = f \circ \phi$ we get

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} \, dxdy$$

$$= \int_{B_1(0) \times B_1(0)} \frac{|g(u) - g(v)|^2}{|\phi(u) - \phi(v)|^{n+1}} \, |\det \nabla \phi(u) \det \nabla \phi(v)| \, dudv$$

$$\geqslant c_{12}^{-(3n+1)} \int_{B_1(0) \times B_1(0)} \frac{|g(u) - g(v)|^2}{|u - v|^{n+1}} \, dudv. \qquad (4.12)$$

An application of Lemma 6 yields

$$\int_{B_1(0) \times B_1(0)} \frac{|g(u) - g(v)|^2}{|u - v|^{n+1}} \, dudv \geqslant c_4 \int_{B_1(0)} \frac{|g(u)|^2}{\rho_{B_1(0)}(u)(1 - (\ln \rho_{B_1(0)}(u))^3)} \, du.$$

Using (4.10) we find (recall $\rho_{B_1(0)}(u) < 1$) that

$$1 - (\ln \rho_{B_1(0)}(u))^3 \leqslant 1 - (\ln c_{12}^{-1} \rho_\Omega(\phi(u)))^3 \leqslant c_{13}(1 + |\ln \rho_\Omega(\phi(u))|^3) \qquad (4.13)$$

for some $c_{13} = c_{13}(\Omega, c_{12}) > 0$. We apply (4.11), (4.13) and then again make the change of variables $x = \phi(u)$ and use (4.7) to get

$$\int\limits_{B_1(0) \times B_1(0)} \frac{|g(u) - g(v)|^2}{|u - v|^{n+1}} \, du dv$$

$$\geqslant \quad c_{12}^{-1} c_{13}^{-1} c_4 \int\limits_{B_1(0)} \frac{|g(u)|^2}{\rho_\Omega(\phi(u))(1 + |\ln \rho_\Omega(\phi(u))|^3)} \, du$$

$$\geqslant \quad c_{12}^{-(n+1)} c_{13}^{-1} c_4 \int\limits_\Omega \frac{|f(x)|^2}{\rho_\Omega(x)(1 + |\ln \rho_\Omega(x)|^3)} \, dx \,. \qquad (4.14)$$

Combining (4.12), (4.14) and letting $c_{14} = c_{12}^{-(4n+2)} c_{13}^{-1} c_4$ we complete the proof. $\square$

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